

Various Modifications to Debye-Hückel Interactions in Solar Equations of State

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Abstract The first order effect of Coulomb forces between the charged particles of a plasma is the well-known Debye-Hückel-term. It is a negative contribution to the pressure and energy of the gas, that at high densities will overwhelm the ideal gas contributions and make the gas implode into a black hole. Nature obviously constrains this term, avoiding this fate, but how? We investigate three different mechanisms and their effects on the equation of state and on solar models, and the physical justifications for each of them. We conclude that higher order Coulomb terms in combination with quantum diffraction of electrons, provide the needed convergence.

Keywords: Atomic processes – Equation of state – Plasmas

1. Introduction and Motivation

The equation of state (EOS) is a fundamental ingredient in any astrophysical modeling, both needed for the thermodynamics of the constituent gas of the object under study, but also as a foundation for opacity calculations, needed for radiative transfer calculations through that same body.

As we now find ourselves in the age of asteroseismology (e.g., Aerts, Christensen-Dalsgaard, and Kurtz 2010; Christensen-Dalsgaard and Houdek 2010; Di Mauro 2016), much higher demands are placed on our models. To meet these demands we must include a much more comprehensive array of realistic physics into our modeling. For the EOS, one such aspect is the thermodynamic impact of Coulomb interactions. The first-order term, derived by (Debye and Hückel 1923), has been included for decades now (Rouse 1962), and since then it has also

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been known to give rise to negative total pressures for high densities and low temperatures.

This is because the first order term is negative and its magnitude increase faster than the other pressure-components. The next term in a density expansion would be positive and prevent a collapse of the plasma by ensuring a positive pressure. It will also diverge for slightly higher densities and temperatures, than before. In other words the expansion is only slowly converging, posing a real problem for high density EOS calculations, such as are needed for low mass stars, planets, white dwarfs, and even the Sun, given the demanding accuracy of helioseismic inversions (as illustrated in the lower right-hand panel of Fig. 2). Remedies for this has been sought, and there are some alternatives to the systematic, but slowly converging expansion in density.

We explore and illustrate some mechanisms that have been used to prevent the negative pressure issue using the example of the MHD EOS (Hummer and Mihalas 1988; Mihalas, Däppen, and Hummer 1988; Däppen et al. 1988), and its descendant the, *T*-MHD EOS, currently in the final stages of development by the authors. All the EOS comparison cases presented here, except for the OPAL EOS (Rogers, Swenson, and Iglesias 1996; Rogers and Nayfonov 2002) have been computed with the *T*-MHD code.

2. The Helmholtz Free Energy and Coulomb Interactions

The *T*-MHD EOS is based on the thermodynamic potential of the *Helmholtz free energy*, F . From this potential, all other thermodynamic variables can be computed as derivatives, e.g.,

$$p = - \left(\frac{\partial F}{\partial V} \right)_{T, \{N\}} , \quad S = - \left(\frac{\partial F}{\partial T} \right)_{V, \{N\}} , \quad (1)$$

for the pressure, p , and the entropy per volume, S , where $\{N\}$ denotes under the constraint of equilibrium densities of all species of particles. These are first-order thermodynamic derivatives. Among the second-order thermodynamic derivatives we find the heat capacities, expansion coefficients, and the adiabatic exponent,

$$\gamma_1 = \left(\frac{\partial \ln p}{\partial \ln \varrho} \right)_{\text{ad}} . \quad (2)$$

For most modern EOS, including the *T*-MHD, both first- and second-order derivatives are analytical, ensuring both thermodynamic consistency, smoothness and enabling stable numerical differentiation of second-order thermodynamic derivatives. It also ensures the quality of the comparisons presented below.

An ideal gas is characterized by ignoring the Coulomb interactions between free charges in the plasma (in spite of such forces being needed for thermalization of the gas by collisions). The effects of including Coulomb interactions increase with the density of such charges. To first order, this is the Debye-Hückel-term (Debye and Hückel 1923),

$$F_{\text{DH}} = -\frac{1}{3} N_{\text{ion}} k_B T \Lambda , \quad (3)$$

for temperature T , and ion number density, N_{ion} . The plasma coupling parameter is,

$$\Lambda = \frac{V}{4\pi N_{\text{ion}}} \lambda_{\text{DH}}^{-3}, \quad (4)$$

where the Debye screening-length, including electron degeneracy via θ_e , is

$$\lambda_{\text{DH}}^{-2} = \frac{4\pi e^2}{k_{\text{B}} T} \left[N_e \theta_e(\eta_e) + \sum_{\alpha \neq e} Z_{\alpha}^2 n_{\alpha} \right]. \quad (5)$$

The non-relativistic version of the degeneracy factor is

$$\theta_e(\eta) = \mathcal{F}_{-1/2}(\eta) / 2\mathcal{F}_{1/2}(\eta) \quad (6)$$

(Cooper and DeWitt 1973), and the degeneracy parameter $\eta = \mu/k_{\text{B}}T$ where μ is the chemical potential of the electrons and $\mathcal{F}_{\nu}(\eta)$ are the Fermi-Dirac integrals.

Since F_{DH} increasingly over-estimates the Coulomb effects with increasing density, and since it provides a negative pressure contribution, F_{DH} needs to be limited by some mechanism, in order to describe a stable, non-collapsing plasma.

We investigate three moderating factors to apply to F_{DH} , for this purpose and, importantly, also to obtain a far better approximation to the plasmas of low-mass stars. The following EOS calculations are performed for a H-He mixture of $X = 71.6\%$ and $Y = 28.4\%$ by mass, respectively (10:1 ratio by number).

We show comparisons between only two of the four independent thermodynamic variables: the pressure, p , which is crucial for the hydrostatic support of stars, and the adiabatic exponent, γ_1 , since it can be inverted for in helioseismic analyses (Basu and Christensen-Dalsgaard 1997; DiMauro et al. 2002). Combined with the fact that below the top 2–3 Mm, the rest of the solar convective envelope is exceedingly close to adiabatic, and its stratification therefore almost entirely determined by the EOS and the composition, makes γ_1 in this region a powerful probe of the EOS.

The T -MHD EOS, used for these calculations, is a descendant of the MHD EOS with many updates to the employed physics. The choices of what physics to use, is controlled by flags, including how F_{DH} should be modified. This flexible approach enables a study like the present, exploring and comparing different choices for isolated parts of the EOS. It will be presented in a future series of papers.

2.1. The τ -correction

The so-called τ -correction was first introduced by Graboske, Harwood, and Rogers (1969, GHR), as a means of truncating the diverging Debye and Hückel (1923)-term, F_{DH} . This factor was subsequently adopted in the MHD EOS by Hummer and Mihalas (1988), and also the Chem EOS by Kilcrease et al. (2015). A discussion of this term can also be found in Trampedach, Däppen, and Baturin (2006).

GHR argued that there would be a minimum distance of approach between ions and electrons of

$$r_{\min} = e^2 \langle Z \rangle \left[k_B T \frac{\mathcal{F}_{3/2}(\eta_e)}{\mathcal{F}_{1/2}(\eta_e)} \right]^{-1}, \quad (7)$$

being the distance at which the kinetic energy of an average electron (the square parenthesis) equals the potential energy around an ion of average charge, $e^2 \langle Z \rangle$. The use of Fermi integrals, \mathcal{F}_ν , allows for arbitrary degeneracy of the electrons. While this r_{\min} could make sense for interactions between ions, it is unclear how this would provide a closest approach between oppositely charged particles.

Proceeding with this r_{\min} , GHR used the formulation developed as part of the original Debye and Hückel (1923) theory of electrolytes, to account for hard sphere behavior of constituent molecules, which results in a factor

$$\tau(x) = 3[\ln(1+x) - x + x^2/2]x^{-3}, \quad x = r_{\min}/\lambda_{\text{DH}}, \quad (8)$$

Ignoring the various degeneracy factors, and assuming electron density, N_e to be proportional to density, ρ , we see that $x \propto \rho^{-1/2} T^{-3/2}$.

The form of the τ -factor arise from the so-called *recharging* procedure (Debye and Hückel 1923), which evaluates the free energy of a potential as an integral of that potential over the charge, from zero to full charge. The procedure is carried out for the screened point-charge potential and the screened, finite volume charge potential, and $\tau(x)$ is the ratio between the two. This recharging process is only valid, however, if r_{\min} does *not* depend on charge, which Eq. (7) plainly does.

The effect of using the $\tau(x)$ -factor on F_{DH} is shown in Fig. 1a). As with all these F_{DH} modifications $\tau(x)$ moderates the negative DH pressure, increasing the total pressure. Of the three modifications we explore here, this one has the largest amplitude, in both pressure, p_{gas} , and adiabatic exponent, γ_1 . The iso-contours of this $\tau(x)$ -effect approximately follow lines with $\rho \propto T^{2.6}$.

2.2. Quantum Diffraction

This is a manifestation of Heisenberg (1927)'s uncertainty principle, effectively turning the classical point charge into an extended distribution of particle with finite charge density, ρ . This has the profound effect that two charged, quantum particles can get arbitrarily close, and instead of diverging, the electrostatic potential between them converges to zero. The result of this is a smooth, short-range truncation of the coulomb interactions, as investigated by, e.g., Riemann et al. (1995). An analytical fit, $q(\gamma_{ee})$, to their results were adopted in the T -MHD EOS (Trampedach and Däppen 2023). The argument to this factor is the de Broglie wavelength between two electrons

$$\lambda_{ee}^2 = \frac{\hbar^2}{k_B T} \frac{\Theta_e}{m_e},$$

in units of the Debye-length, Eq. (5), $\gamma_{ee} = \lambda_{ee}/\lambda_{\text{DH}}$. The factor, Θ_e , corrects for electron degeneracy. Ignoring, for the moment, θ_e and Θ_e we find that $\gamma_{ee} \propto \sqrt{\rho}/T$.

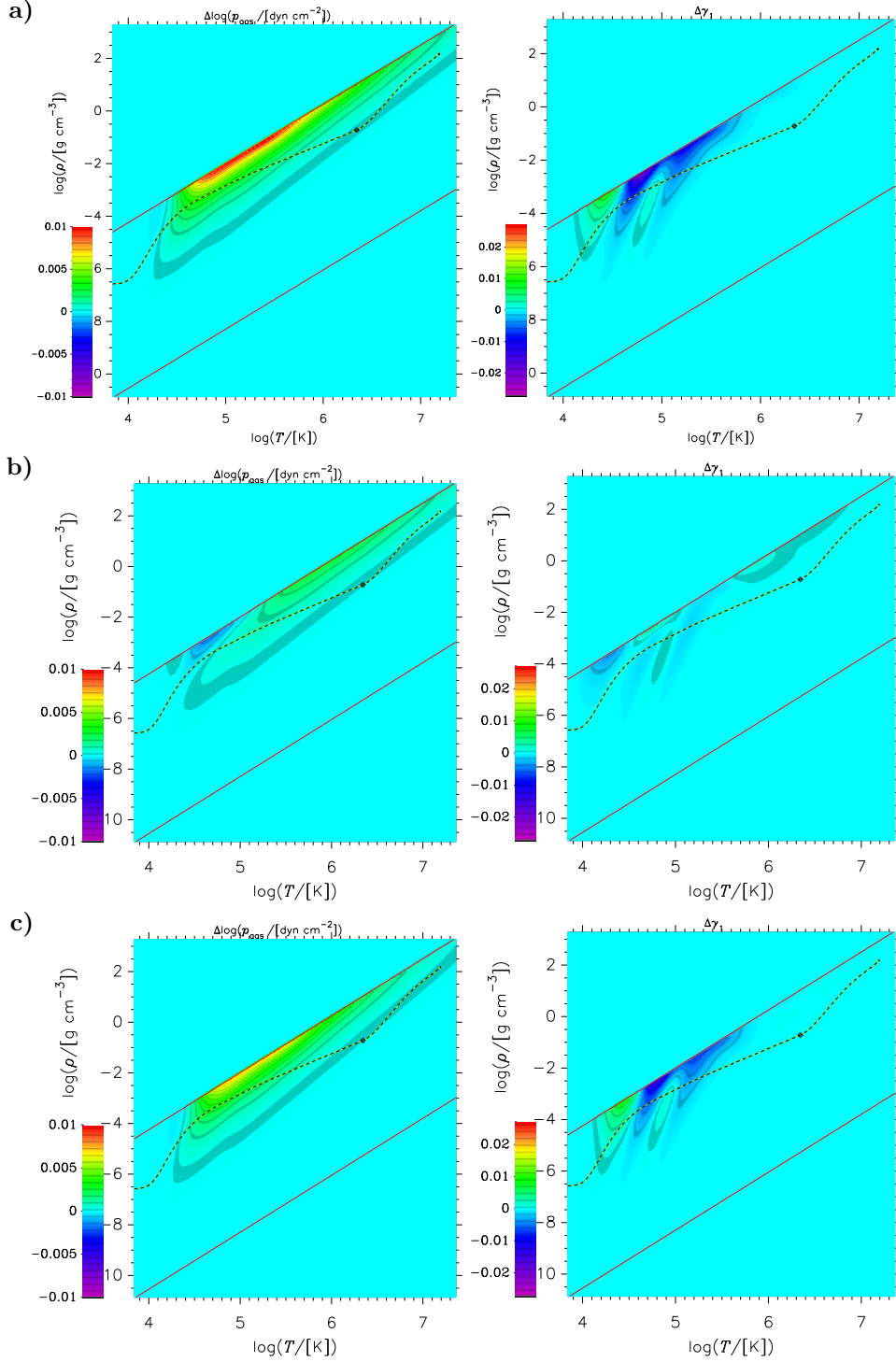


Figure 1. The shading shows the differences between different modifications of the DH-term in the three rows, in the sense $F_{\text{DH}} \times [\{1, \tau(x), q(\gamma_{ee})\} - g(\Lambda)q(\gamma_{ee})]$, comparing three cases against the $g(\Lambda)q(\gamma_{ee})$ -case used for T -MHD. Turquoise shows the zero-level in all panels, but there are separate scales for the left and right columns, as shown with the color-bars for each panel. **a)** the bare F_{DH} , **b)** $\tau(x)$ (see Sect. 2.1), **c)** quantum diffraction, only (see Sect. 2.2), effectively showing the consequences of higher-order Coulomb terms, $g(\Lambda)$ (see Sect. 2.3). The left column shows gas pressure differences, and the right column shows γ_1 differences. The dashed line shows the stratification of solar ModelS by Christensen-Dalsgaard et al. (1996). The inflection around $\log T = 6.4$, marked with a small \diamond -symbol, is the bottom of the convective envelope, where the temperature gradient changes from adiabatic to radiative. Diagonal lines show the extent of the tables.

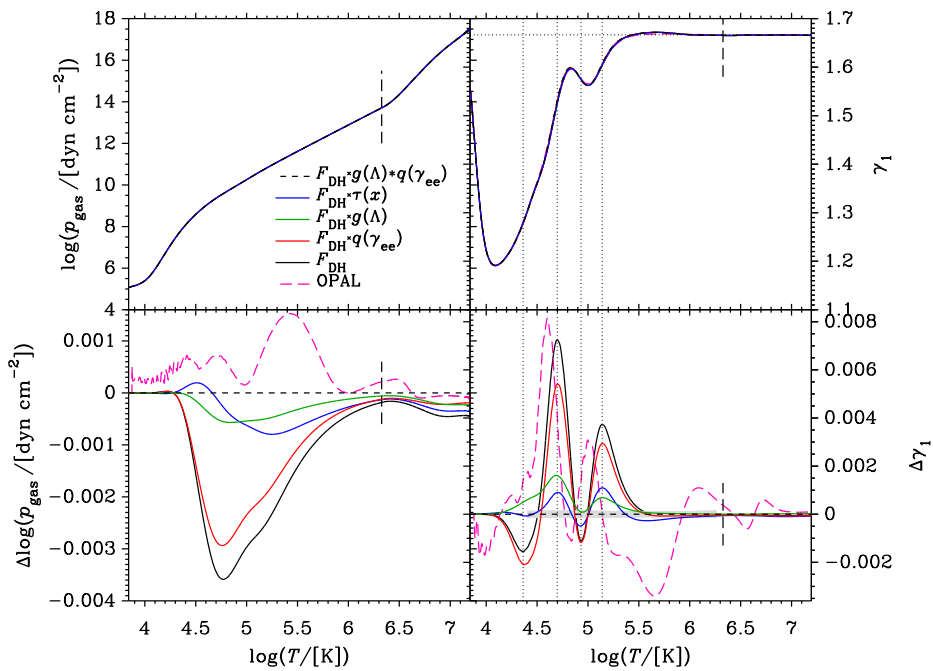


Figure 2. The differences shown in Fig. 1, but for the solar stratification, also shown in Fig. 1. The various cases, X as list in the top-left panel, are all plotted in the top two plots of absolute $\log p_g$ and γ_1 , but they can only be distinguished at high magnification. The dotted horizontal line in the top-right plot shows the non-ionizing perfect gas value of $\gamma_1 = 5/3$. The bottom two plots shows the effects of applying the $\tau(x)$, $g(\Lambda)$, quantum diffraction and the combined Coulomb and quantum diffraction factors to the Debye and Hückel (1923)-term, in the sense (case X minus T -MHD). The long-dashed magenta lines show the OPAL EOS for comparison (Rogers and Nayfonov 2002). The high-frequency noise at low T is interpolation errors from the OPAL routines. The vertical dotted lines in the right-hand-side plots show the location of extrema in $\Delta\gamma_1$. The vertical, long-dashed line-segments show the location of the bottom of the convective envelope. The gray band around the zero-line of the $\Delta\gamma_1$ plot shows the error-bars from a helioseismic inversion of γ_1 , by Di Mauro et al. (2002), highlighting how observationally significant the EOS differences are.

This factor goes from 1 to 0, asymptotically going to zero as $\gamma_{ee}^{-1/2}$ for large γ_{ee} , but applies only to the electron-electron part of F_{DH} , which we indicate with an asterisk, $*$, instead of an ordinary multiplication. The effect of electron-electron diffraction is shown for the tables in Fig. 1 as the difference between rows a-c and iso-contours are found to approximately follow $\rho \propto T^{2.2}$, which is shallower than for $\tau(x)$, Sect. 2.1. The effect on the solar case is shown in Fig. 2, as the difference between the black and red solid curves, and is only about a third of the other two cases, $\tau(x)$: difference between solid black and blue, $g(\Lambda)$: solid black and green, both for the pressure and the adiabatic exponent.

For the interaction between identical particles, there is an additional quantum mechanical term describing the exchange effect, F_{ex} , which is a consequence of Pauli (1925)'s exclusion principle; If two identical fermions meet they will either form a symmetric wave-function (when they form an anti-symmetric singlet spin state) or an anti-symmetric wave-function (when they form a symmetric

triplet spin state). The latter case will prevent any overlap between the two particles, thereby further reducing the interaction energy between them. This is not formulated as a factor on the Debye-Hückel term, and although it is included in all the T -MHD calculations shown here, it is not addressed in the present analysis.

2.3. Higher-order Coulomb interactions

The long-range aspect of Coulomb forces greatly complicates the treatment of Coulomb interacting systems. Only crystalline structures can be handled analytically in closed form, due to their spatially periodic structure. For gases and the high-density transition to a liquid, there is no such simplifying mechanism. That leaves us with two tools for investigating these systems. Analytical expansions in density (or equivalent quantity), or simulations (e.g., Monte Carlo or Molecular Dynamics). Analytical expansions can be derived rigorously and include all the physics to a given order, but the complexity rises exponentially with order, quickly making this approach prohibitively labor intensive. So far expansions up to order $\rho^{5/2}$ have been accomplished (DeWitt et al. 1995; Alastuey and Perez 1996) and are available to astronomers (Rogers, Swenson, and Iglesias 1996; Rogers and Nayfonov 2002). Despite this effort, an order 5/2 in density is unlikely to suffice for low mass red dwarfs and brown dwarfs, and will then diverge rapidly. The simulation alternative can be applied to any conditions needed, but are instead limited by the computational cost to run them with enough particles to make statistically accurate results, and avoid boundary effects. Many such simulations are usually needed in order to populate parameter space to strongly constrain fits to the effect under investigation. We have chosen the simulation approach in order to obtain an EOS with as wide a region of applicability/validity, as possible. We employ the Monte Carlo simulations by Slattery, Doolen, and DeWitt (1982) and Stringfellow, DeWitt, and Slattery (1990), to which we have fitted our own expression for the Coulomb factor, $g(\Lambda)$, also constrained by the low density (Abe 1959) cluster-expansion. Λ is the plasma coupling parameter. At small Λ , $g \propto -\Lambda$ and for Λ approaching the crystallization limit of $\Lambda_{\text{cr}} = 4113$, the Monte Carlo simulations give $g \propto \Lambda^{-0.31540}$.

The effect of omitting our implementation of $g(\Lambda)$, is shown in Fig. 1c). The effect of omitting the combined effects of quantum diffraction and higher order Coulomb terms, $q(\gamma_{\text{ee}}) * g(\Lambda) = T\text{-MHD}$, is shown in Fig. 1a). The difference, $\tau(x) - q(\gamma_{\text{ee}}) * g(\Lambda)$ of Fig. 1b), shows that $\tau(x)$ generally gives higher pressure than $T\text{-MHD}$, with increasing density, and the transition is smoother in density with $g(\Lambda)$ than with $\tau(x)$. Iso-contours of the effect of $g(\Lambda)$ approximately follows $\rho \propto T^{2.6}$ just as with $\tau(x)$, and is steeper than for quantum diffraction, Sect. 2.2.

3. Discussion and Conclusion

We have investigated three different modifications to the first-order Coulomb term of the Helmholtz free energy. The first one, the $\tau(x)$ factor, purports to

model the effects of a limit to how close ions and electrons can approach each other at a given set of plasma conditions. Both the argument and the derivation are found to be flawed, however.

The second modification is the quantum diffraction, which describes the quantum-mechanical analogy to a closest approach, provided by Heisenberg’s uncertainty relation, turning classical point-particles into charge distributions of finite size. As, e.g., electrons overlap, they feel less and less charge interior to their location, due to the everywhere finite charge density. The effect of this is only about a third of the effect from $\tau(x)$, and has a different shape. It is also a physical effect that needs to be included in the EOS if aiming for high-fidelity models of low-mass stellar objects, or comparing against helioseismology.

The last modification is that due to the higher-order terms in the Coulomb interactions, in this case found from Monte Carlo simulations. The effect of this is curiously similar to that of $\tau(x)$, although less steep in density, and of slightly lower amplitude. This factor is also a physical effect and it needs to be combined with quantum diffraction as shown in Fig. 2 for the solar case. This combination has a slightly *larger* amplitude in a solar model, than does the $\tau(x)$ factor.

The effect on the adiabatic exponent, γ_1 , is very similar in shape for all three cases, as seen in Fig. 2, except quantum diffraction which only *reduces* γ_1 . From the vertical dotted lines in the γ_1 panels of Fig. 2, it is evident that the differences do not constitute a simple deepening of the ionization dips of γ_1 , but is rather a tilting of the dips, to have lower ‘background’ on the cool side and very slightly higher ‘background’ on the warm side, as well as a shift of the ionization dips towards higher T . These changes to γ_1 will affect helioseismic abundance determinations which exploit these ionization dips to determine helium (Däppen and Gough 1984; Antia and Basu 1994; Basu and Antia 1995; Richard et al. 1998; Vorontsov, Baturin, and Pamiatnykh 1991; Houdek and Gough 2007) and metal abundances (Lin and Däppen 2005; Houdek and Gough 2011; Vorontsov et al. 2014; Buldgen et al. 2017) in the close to adiabatic part of the convective envelope. This is the only method available for a solar He determination, and an important test of spectral abundance analysis for metals.

Trampedach, Däppen, and Baturin (2006) compared the OPAL and MHD EOS and found that the $\tau(x)$ factor, used in the MHD EOS, was reducing F_{DH} by more than the higher-order effects in the OPAL EOS, which are based on analytical expansions. Because the OPAL EOS was found to be in better general agreement with helioseismology, Trampedach, Däppen, and Baturin (2006) assumed the $\tau(x)$ factor was overestimating the modification of F_{DH} . Based on our present results (lower-left panel of Fig. 2, however, it is possible that the MHD EOS, with the τ -factor, had the better implementation of Coulomb interactions, as the differences with respect to the T -MHD case of $g(\Lambda) * q(\gamma_{\text{ee}})$, is quite small. Such a conclusion assumes that T -MHD is closer to reality than OPAL, and that the many other differences between the two EOS have smaller effects. This is a possibility that warrants further investigation.

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References

- Abe, R.: 1959, Giant Cluster Expansion Theory and Its Application to High Temperature Plasma. *Prog. Theor. Phys.* **22**, 213.
- Aerts, C., Christensen-Dalsgaard, J., Kurtz, D.W.: 2010, *Asteroseismology*, 1st edn., *Astron. & Astroph. Library*, Springer, Dordrecht.
- Alastuey, A., Perez, A.: 1996, Virial Expansion for Quantum Plasmas: Fermi-Bose statistics. *Phys. Rev. E* **53**, 5714. [ADS](#).
- Antia, H.M., Basu, S.: 1994, Measuring the helium abundance in the solar envelope: The role of the equation of state. *ApJ* **426**, 801. [ADS](#).
- Basu, S., Antia, H.M.: 1995, Helium abundance in the solar envelope. *MNRAS* **276**, 1402. [ADS](#).
- Basu, S., Christensen-Dalsgaard, J.: 1997, Equation of state and helioseismic inversions. *A&A* **322**, L5. [ADS](#).
- Buldgen, G., Salmon, S.J.A.J., Noels, A., Scuflaire, R., Dupret, M.A., Reese, D.R.: 2017, Determining the metallicity of the solar envelope using seismic inversion techniques. *MNRAS* **472**, 751. [ADS](#).
- Christensen-Dalsgaard, J., Houdek, G.: 2010, Prospects for asteroseismology. *Ap&SS* **328**, 51. [ADS](#).
- Christensen-Dalsgaard, J., Däppen, W., Ajukov, S.V., Anderson, E.R., Antia, H.M., Basu, S., Baturin, V.A., Berthomieu, G., Chaboyer, B., Chitre, S.M., Cox, A.N., Demarque, P., Donatowicz, J., Dziembowski, W.A., Gabriel, M., Gough, D.O., Guenther, D.B., Guzik, J.A., Harvey, J.W., Hill, F., Houdek, G., Iglesias, C.A., Kosovichev, A.G., Leibacher, J.W., Morel, P., Proffitt, C.R., Provost, J., Reiter, J., Rhodes Jr., E.J., Rogers, F.J., Roxburgh, I.W., Thompson, M.J., Ulrich, R.K.: 1996, The Current State of Solar Modeling. *Science* **272**, 1286. [ADS](#).
- Cooper, M.S., DeWitt, H.E.: 1973, Degeneracy effects in gases in the near-classical limit. *Phys. Rev. A* **8**, 1910. [ADS](#).
- Däppen, W., Gough, D.O.: 1984, On the Determination of the Helium Abundance of the Solar Convection Zone. In: Ledoux, P. (ed.) *Theoretical Problems in Stellar Stability and Oscillations*, *Liège Int. Astroph. Coll.*, 264. [ADS](#).
- Däppen, W., Mihalas, D., Hummer, D.G., Mihalas, B.W.: 1988, The equation of state for stellar envelopes. III. Thermodynamic Quantities. *ApJ* **332**, 261. [ADS](#).
- Debye, P., Hückel, E.: 1923, Zur Theorie der Electrolyte. *Physic. Zeit.* **24**, 185.
- DeWitt, H.E., Schlages, M., Sakakura, A.Y., Kraeft, W.D.: 1995, Low density expansion of the equation of state for a quantum electron gas. *Phys. Lett. A* **30**, 326.
- Di Mauro, M.: 2016, A review on Asteroseismology. In: *Frontier Research in Astrophysics – II*, 29:1. [ADS](#).
- Di Mauro, M., Christensen-Dalsgaard, J., Rabello-Soares, M.C., Basu, S.: 2002, Inferences on the Solar envelope with high degree-modes. *A&A* **384**, 666. [ADS](#).
- Graboske, H.C., Harwood, D.J., Rogers, F.J.: 1969, Thermodynamic Properties of Nonideal gases. I. Free-Energy Minimization Method. *Physical Review* **186**, 210. [ADS](#).
- Heisenberg, W.: 1927, Über die Grundprinzipien der Quantenmechanik. *Forsch. und Fortschr.* **3**, 83.
- Houdek, G., Gough, D.O.: 2007, An asteroseismic signature of helium ionization. *MNRAS* **375**, 861. [ADS](#).
- Houdek, G., Gough, D.O.: 2011, On the seismic age and heavy-element abundance of the Sun. *MNRAS* **418**, 1217. [ADS](#).
- Hummer, D.G., Mihalas, D.: 1988, The equation of state for stellar envelopes. I. An occupation probability formalism for the truncation of internal partition functions. *ApJ* **331**, 794. [ADS](#).
- Kilcrease, D.P., Colgan, J., Hakel, P., Fontes, C.J., Sherrill, M.E.: 2015, An equation of state for partially ionized plasmas: The Coulomb contribution to the free energy. *High Energy Density Phys.* **16**, 36. [ADS](#).

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- Lin, C.-H., Däppen, W.: 2005, The Chemical Composition and Equation of State of the Sun Inferred from Seismic Models through an Inversion Procedure. *ApJ* **623**, 556. [ADS](#).
- Mihalas, D., Däppen, W., Hummer, D.G.: 1988, The equation of state for stellar envelopes. II. Algorithm and Selected Results. *ApJ* **331**, 815. [ADS](#).
- Pauli, W.: 1925, Über den Zusammenhang des Abschlusses der Elektronengruppen im Atom mit der Komplexstruktur der Spektren. *Zeits. f. Physik* **31**, 765. [ADS](#).
- Richard, O., Dziembowski, W.A., Sienkiewicz, R., Goode, P.R.: 1998, Precise Determination of the Solar Helium Abundance by Helioseismology. In: Korzennik, S.G. (ed.) *Structure and Dynamics of the Interior of the Sun and Sun-like Stars, SOHO 6/GONG 98 Workshop*, ESA, Noordwijk, 517. [ADS](#).
- Riemann, J., Schlages, M., DeWitt, H.E., Kraeft, W.D.: 1995, Equation of state of the weakly degenerate one-component plasma. *Physica A* **219**, 423. [ADS](#).
- Rogers, F.J., Nayfonov, A.: 2002, Updated and expanded OPAL equation of state tables: Implications for helioseismology. *ApJ* **576**, 1064. [ADS](#).
- Rogers, F.J., Swenson, F.J., Iglesias, C.A.: 1996, OPAL Equation-of-State Tables for Astrophysical Applications. *ApJ* **456**, 902. [ADS](#).
- Rouse, C.: 1962, Ionization-Equilibrium Equation of State. III. Results with Debye-Hückel Corrections and Planck's Partition Function. *ApJ* **136**, 636. [ADS](#).
- Slattery, W.L., Doolen, G.D., DeWitt, H.E.: 1982, N -dependence in the classical one-component plasma Monte Carlo calculations. *Phys. Rev. A* **26**, 2255. [ADS](#).
- Stringfellow, G.S., DeWitt, H.E., Slattery, W.L.: 1990, Equation of state of the one-component plasma derived from precision Monte Carlo calculations. *Phys. Rev. A* **41**, 1105. [ADS](#).
- Trampedach, R., Däppen, W.: 2023, T -MHD – An EOS for Whole Stars. I. Including all the physics. *ApJ*. (in preparation).
- Trampedach, R., Däppen, W., Baturin, V.A.: 2006, A synoptic comparison of the Mihalas-Hummer-Däppen and OPAL equations of state. *ApJ* **646**, 560. [ADS](#).
- Vorontsov, S.V., Baturin, V.A., Pamiatnykh, A.A.: 1991, Seismological measurement of solar helium abundance. *Nature* **349**, 49. [ADS](#).
- Vorontsov, S., Baturin, V., Ayukov, S., Gryaznov, V.: 2014, Helioseismic measurements in the solar envelope using group velocities of surface waves. *Mon. Not. R. Astron. Soc.* **441**, 3296. [ADS](#).